CHAPTER 4

Section 4.1

1.

a. P(x ≤ 1) = $\int_{-\infty}^{1} f(x)dx = \int_0^1 \frac{1}{2}xdx = \frac{1}{4}x^2\Big]_0^1 = .25$

b. P(.5 ≤ X ≤ 1.5) = $\int_{.5}^{1.5} \frac{1}{2}xdx = \frac{1}{4}x^2\Big]_{.5}^{1.5} = .5$

c. P(x > 1.5) = $\int_{1.5}^{\infty} f(x)dx = \int_{1.5}^{2} \frac{1}{2}xdx = \frac{1}{4}x^2\Big]_{1.5}^{2} = \frac{7}{16} \approx .438$

2. F(x) = $\frac{1}{10}$ for –5 ≤ x ≤ 5, and = 0 otherwise

a. P(X < 0) = $\int_{-5}^{0} \frac{1}{10}dx = .5$

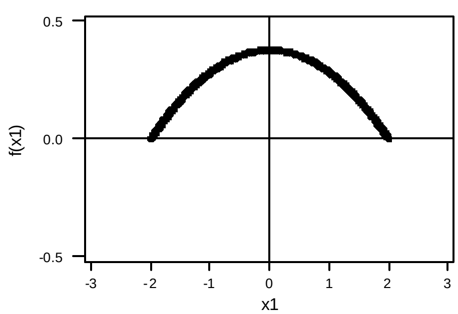
b. P(-2.5 < X < 2.5) = $\int_{-2.5}^{2.5} \frac{1}{10}dx = .5$

c. P(-2 ≤ X ≤ 3) = $\int_{-2}^{3} \frac{1}{10}dx = .5$

d. P(k < X < k + 4) = $\int_{k}^{k+4} \frac{1}{10}dx = \frac{x}{10}\Big]_k^{k+4} = \frac{1}{10}[(k+4)-k] = .4$

3.

a. Graph of f(x) = .09375(4 – x^2)



b. P(X > 0) = $\int_{0}^{2} .09375(4 - x^2)dx = .09375(4x - \frac{x^3}{3})\Big]_0^2 = .5$

c. P(-1 < X < 1) = $\int_{-1}^{1} .09375(4 - x^2)dx = .6875$

d. P(x < -.5 OR x > .5) = 1 – P(-.5 ≤ X ≤ .5) = 1 - $\int_{-.5}^{.5} .09375(4 - x^2)dx$
= 1 - .3672 = .6328

4.

a. $\int_{-\infty}^{\infty} f(x;q)dx = \int_0^{\infty} \frac{x}{q^2} e^{-x^2/2q^2} dx = -e^{-x^2/2q^2}\Big]_0^{\infty} = 0 - (-1) = 1$

b. P(X ≤ 200) = $\int_{-\infty}^{200} f(x;q)dx = \int_0^{200} \frac{x}{q^2} e^{-x^2/2q^2} dx$

$= -e^{-x^2/2q^2}\Big]_0^{200} \approx -.1353 + 1 = .8647$

P(X < 200) = P(X ≤ 200) ≈ .8647, since x is continuous.
P(X ≥ 200) = 1 - P(X ≤ 200) ≈ .1353

c. P(100 ≤ X ≤ 200) = $\int_{100}^{200} f(x;q)dx = -e^{-x^2/20,000}\Big]_{100}^{200} \approx .4712$

d. For x > 0, P(X ≤ x) =
$\int_{-\infty}^{x} f(y;q)dy = \int_0^{x} \frac{y}{e^2} e^{-y^2/2q^2} dx = -e^{-y^2/2q^2}\Big]_0^{x} = 1 - e^{-x^2/2q^2}$

5.

a. 1 = $\int_{-\infty}^{\infty} f(x)dx = \int_0^2 kx^2 dx = k\left(\frac{x^3}{3}\right)\Big]_0^2 = k\left(\frac{8}{3}\right) \Rightarrow k = \frac{3}{8}$

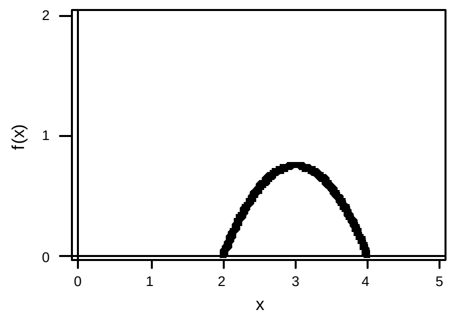
b. P(0 ≤ X ≤ 1) = $\int_0^1 \frac{3}{8}x^2 dx = \frac{1}{8}x^3\Big]_0^1 = \frac{1}{8} = .125$

c. P(1 ≤ X ≤ 1.5) = $\int_1^{1.5} \frac{3}{8}x^2 dx = \frac{1}{8}x^3\Big]_1^{1.5} = \frac{1}{8}\left(\frac{3}{2}\right)^3 - \frac{1}{8}(1)^3 = \frac{19}{64} \approx .2969$

d. P(X ≥ 1.5) = 1 - $\int_0^{1.5} \frac{3}{8}x^2 dx = \frac{1}{8}x^3\Big]_0^{1.5} = 1 - \left[\frac{1}{8}\left(\frac{3}{2}\right)^3 - 0\right] = 1 - \frac{27}{64} = \frac{37}{64} \approx .5781$

6.

a.



b. 1 = $\int_2^4 k[1-(x-3)^2]dx = \int_{-1}^1 k[1-u^2]du = \frac{4}{3} \Rightarrow k = \frac{3}{4}$

c. P(X > 3) = $\int_3^4 \frac{3}{4}[1-(x-3)^2]dx = .5$ by symmetry of the p.d.f

d. $P\left(\frac{11}{4} \le X \le \frac{13}{4}\right) = \int_{11/4}^{13/4} \frac{3}{4}[1-(x-3)^2]dx = \frac{3}{4}\int_{-1/4}^{1/4}[1-(u)^2]du = \frac{47}{128} \approx .367$

e. P(|X-3| > .5) = 1 – P(|X-3| ≤ .5) = 1 – P(2.5 ≤ X ≤ 3.5)

= 1 - $\int_{-.5}^{.5} \frac{3}{4}[1-(u)^2]du = \frac{5}{16} \approx .313$

7.

a. f(x) = $\frac{1}{10}$ for 25 ≤ x ≤ 35 and = 0 otherwise

b. P(X > 33) = $\int_{33}^{35} \frac{1}{10} dx = .2$

c. E(X) = $\int_{25}^{35} x \cdot \frac{1}{10} dx = \left.\frac{x^2}{20}\right]_{25}^{35} = 30$

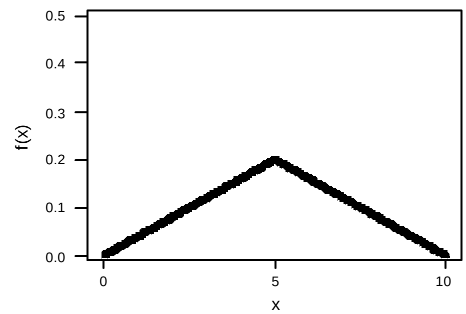
30 ± 2 is from 28 to 32 minutes:

P(28 < X < 32) = $\int_{28}^{32} \frac{1}{10} dx = \left.\frac{1}{10}x\right]_{28}^{32} = .4$

d. P(a ≤ x ≤ a+2) = $\int_a^{a+2} \frac{1}{10} dx = .2$, since the interval has length 2.

8.

a.



b. $\int_{-\infty}^{\infty} f(y)dy = \int_0^5 \frac{1}{25} y dy + \int_5^{10} (\frac{2}{5} - \frac{1}{25} y) dy = \left. \frac{y^2}{50} \right]_0^5 + \left. \left(\frac{2}{5} y - \frac{1}{50} y^2 \right) \right]_5^{10}$

$= \frac{1}{2} + \left[(4-2) - (2 - \frac{1}{2}) \right] = \frac{1}{2} + \frac{1}{2} = 1$

c. P(Y ≤ 3) = $\int_0^3 \frac{1}{25} y dy = \left. \frac{y^2}{50} \right]_0^5 = \frac{9}{50} \approx .18$

d. P(Y ≤ 8) = $= \int_0^5 \frac{1}{25} y dy + \int_5^8 (\frac{2}{5} - \frac{1}{25} y) dy = \frac{23}{25} \approx .92$

e. P(3 ≤ Y ≤ 8) = P(Y ≤ 8) - P(Y < 3) = $\frac{46}{50} - \frac{9}{50} = \frac{37}{50} = .74$

f. P(Y < 2 or Y > 6) = $= \int_0^3 \frac{1}{25} y dy + \int_6^{10} (\frac{2}{5} - \frac{1}{25} y) dy = \frac{2}{5} = .4$

9.

a. P(X ≤ 6) = $= \int_{.5}^6 .15 e^{-.15(x-5)} dx = .15 \int_0^{5.5} e^{-.15u} du$ (after u = x - .5)

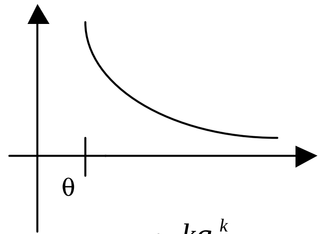
$= \left. e^{-.15u} \right]_0^{5.5} = 1 - e^{-.825} \approx .562$

b. 1 - .562 = .438; .438

c. P(5 ≤ Y ≤ 6) = P(Y ≤ 6) - P(Y ≤ 5) ≈ .562 - .491 = .071

10.

a.



b. $= \int_{-\infty}^{\infty} f(x;k,\theta)dx = \int_{\theta}^{\infty} \frac{k\theta^k}{x^{k+1}}dx = \theta^k \cdot \left(-\frac{1}{x^k}\right)\Big]_{\theta}^{\infty} = \frac{\theta^k}{\theta^k} = 1$

c. P(X ≤ b) = $\int_{\theta}^{b} \frac{k\theta^k}{x^{k+1}}dx = \theta^k \cdot \left(-\frac{1}{x^k}\right)\Big]_{\theta}^{b} = 1 - \left(\frac{\theta}{b}\right)^k$

d. P(a ≤ X ≤ b) = $\int_{a}^{b} \frac{k\theta^k}{x^{k+1}}dx = \theta^k \cdot \left(-\frac{1}{x^k}\right)\Big]_{a}^{b} = \left(\frac{\theta}{a}\right)^k - \left(\frac{\theta}{b}\right)^k$

Section 4.2

11.

a. P(X ≤ 1) = F(1) = $\frac{1}{4} = .25$

b. P(.5 ≤ X ≤ 1) = F(1) – F(.5) = $\frac{3}{16} = .1875$

c. P(X > .5) = 1 – P(X ≤ .5) = 1 – F(.5) = $\frac{15}{16} = .9375$

d. .5 = $F(\tilde{\mu}) = \frac{\tilde{\mu}^2}{4} \Rightarrow \tilde{\mu}^2 = 2 \Rightarrow \tilde{\mu} = \sqrt{2} \approx 1.414$

e. f(x) = F′(x) = $\frac{x}{2}$ for 0 ≤ x < 2, and = 0 otherwise

f. E(X) = $\int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^2 x \cdot \frac{1}{2}xdx = \frac{1}{2}\int_0^2 x^2dx = \frac{x^3}{6}\Big]_0^2 = \frac{8}{6} \approx 1.333$

g. E(X^2) = $\int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^2 x^2 \frac{1}{2}xdx = \frac{1}{2}\int_0^2 x^3dx = \frac{x^4}{8}\Big]_0^2 = 2,$

So Var(X) = E(X^2) – [E(X)]^2 = $2 - \left(\frac{8}{6}\right)^2 = \frac{8}{36} \approx .222$, $\sigma_x \approx .471$

h. From **g**, E(X^2) = 2

12.

a. P(X < 0) = F(0) = .5

b. P(-1 ≤ X ≤ 1) = F(1) – F(-1) = $\frac{11}{16} = .6875$

c. P(X > .5) = 1 – P(X ≤ .5) = 1 – F(.5) = 1 - .6836 = .3164

d. F(x) = F′(x) = $\frac{d}{dx}\left(\frac{1}{2}+\frac{3}{32}\left(4x-\frac{x^3}{3}\right)\right) = 0+\frac{3}{32}\left(4-\frac{3x^2}{3}\right) = .09375\left(4-x^2\right)$

e. $F(\tilde{m}) = .5$ by definition. F(0) = .5 from **a** above, which is as desired.

13.

a. $1=\int_1^\infty \frac{k}{x^4}dx \Rightarrow 1=\frac{-k}{3}x^{-3}\Big|_1^\infty \Rightarrow 1=0-(-\frac{k}{3})(1)\Rightarrow 1=\frac{k}{3}\Rightarrow k=3$

b. cdf: F(x)= $\int_{-\infty}^x f(y)dy = \int_1^x 3y^{-4}dy = -\frac{3}{3}y^{-3}\Big|_1^x = -x^{-3}+1 = 1-\frac{1}{x^3}$. So

$$F(x)=\begin{cases} 0, & x\le 1 \\ 1-x^{-3}, & x>1 \end{cases}$$

c. P(x > 2) = 1 – F(2) = 1 – $\left(1-\frac{1}{8}\right)=\frac{1}{8}$ or .125;

$$P(2<x<3)=F(3)-F(2)=\left(1-\tfrac{1}{27}\right)-\left(1-\tfrac{1}{8}\right)=.963-.875=.088$$

d. $E(x)=\int_1^\infty x\left(\frac{3}{x^4}\right)dx=\int_1^\infty\left(\frac{3}{x^3}\right)dx=-\frac{3}{2}x^{-2}\Big|_1^x = 0+\frac{3}{2}=\frac{3}{2}$

$$E(x^2)=\int_1^\infty x^2\left(\frac{3}{x^4}\right)dx=\int_1^\infty\left(\frac{3}{x^2}\right)dx=-3x^{-1}\Big|_1^x = 0+3=3$$

$$V(x)=E(x^2)-[E(x)]^2=3-\left(\frac{3}{2}\right)^2=3-\frac{9}{4}=\frac{3}{4}$$ or .75

$$s=\sqrt{V(x)}=\sqrt{3/4}=.866$$

e. $P(1.5-.866<x<1.5+.866)=P(x<2.366)=F(2.366)$

$$=1-(2.366^{-3})=.9245$$

14.

a. If X is uniformly distributed on the interval from A to B, then

$$E(X)=\int_A^B x\cdot\frac{1}{B-A}\,dx=\frac{A+B}{2},\ E(X^2)=\frac{A^2+AB+B^2}{3}$$

$V(X) = E(X^2) - [E(X)]^2 = \dfrac{(B-A)^2}{2}$.

With A = 7.5 and B = 20, E(X) = 13.75, V(X) = 13.02

b. $F(X)=\begin{cases}0 & x<7.5\\ \dfrac{x-7.5}{12.5} & 7.5\le x<20\\ 1 & x\ge 20\end{cases}$

c. $P(X \le 10) = F(10) = .200$; $P(10 \le X \le 15) = F(15) - F(10) = .4$

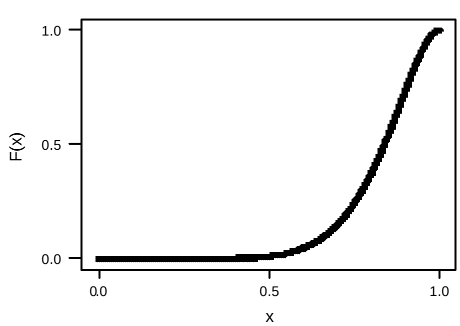
d. $\sigma = 3.61$, so $\mu \pm \sigma = (10.14, 17.36)$
Thus, $P(\mu - \sigma \le X \le \mu + \sigma) = F(17.36) - F(10.14) = .5776$
Similarly, $P(\mu - \sigma \le X \le \mu + \sigma) = P(6.53 \le X \le 20.97) = 1$

15.

a. $F(X) = 0$ for $x \le 0$, $= 1$ for $x \ge 1$, and for $0 < X < 1$,

$$F(X)=\int_{-\infty}^{x} f(y)dy=\int_0^x 90y^8(1-y)dy=90\int_0^x (y^8-y^9)dy$$

$$90\left(\tfrac{1}{9}y^9-\tfrac{1}{10}y^{10}\right)\Big]_0^x=10x^9-9x^{10}$$



b. $F(.5) = 10(.5)^9 - 9(.5)^{10} \approx .0107$

c. $P(.25 \le X \le .5) = F(.5) - F(.25) \approx .0107 - [10(.25)^9 - 9(.25)^{10}]$
$\approx .0107 - .0000 \approx .0107$

d. The 75th percentile is the value of x for which F(x) = .75
$\Rightarrow .75 = 10(x)^9 - 9(x)^{10}$ $\Rightarrow x \approx .9036$

e. E(X) = $\int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^1 x \cdot 90x^8(1-x)dx = 90\int_0^1 x^9(1-x)dx$

$= 9x^{10} - \frac{90}{11}x^{11}\Big]_0^1 = \frac{9}{11} \approx .8182$

E(X^2) = $\int_{-\infty}^{\infty} x^2 \cdot f(x)dx = \int_0^1 x^2 \cdot 90x^8(1-x)dx = 90\int_0^1 x^{10}(1-x)dx$

$= \frac{90}{11}x^{11} - \frac{90}{12}x^{12}\Big]_0^1 \approx .6818$

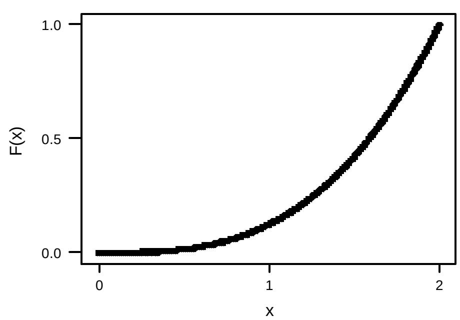
V(X) ≈ .6818 – $(.8182)^2$ = .0124, σ_x = .11134.

f. $\mu \pm \sigma$ = (.7068, .9295). Thus, P($\mu - \sigma \le X \le \mu + \sigma$) = F(.9295) – F(.7068)
= .8465 - .1602 = .6863

16.

a. F(x) = 0 for x < 0 and F(x) = 1 for x > 2. For $0 \le x \le 2$,

F(x) = $\int_0^x \frac{3}{8}y^2dy = \frac{1}{8}y^3\Big]_0^x = \frac{1}{8}x^3$



b. P(x ≤ .5) = F(.5) = $\frac{1}{8}\left(\frac{1}{2}\right)^3 = \frac{1}{64}$

c. P(.25 ≤ X ≤ .5) = F(.5) – F(.25) $= \frac{1}{64} - \frac{1}{8}\left(\frac{1}{4}\right)^3 = \frac{7}{512} \approx .0137$

d. .75 = F(x) = $\frac{1}{8}x^3 \Rightarrow x^3 = 6 \Rightarrow x \approx 1.8171$

e. E(X) = $\int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^2 x \cdot \left(\frac{3}{8}x^2\right)dx = \frac{3}{8}\int_0^1 x^3dx = \frac{3}{8}\left(\frac{1}{4}x^4\right)\Big]_0^2 = \frac{3}{2} = 1.5$

E(X^2) = $\int_0^2 x \cdot \left(\frac{3}{8}x^2\right)dx = \frac{3}{8}\int_0^1 x^4dx = \frac{3}{8}\left(\frac{1}{5}x5\right)\Big]_0^2 = \frac{12}{5} = 2.4$

V(X) = $\frac{12}{5} - \left(\frac{3}{2}\right)^2 = \frac{3}{20} = .15$ σ_x = .3873

f. $\mu \pm \sigma$ = (1.1127, 1.8873). Thus, P($\mu - \sigma \le X \le \mu + \sigma$) = F(1.8873) – F(1.1127) = .8403 - .1722 = .6681

17.

a. For $2 \le X \le 4$, $F(X) = \int_{-\infty}^{x} f(y)dy = \int_{2}^{x} \frac{3}{4}[1-(y-3)^2]dy$ (let u = y-3)

$= \int_{-1}^{x-3} \frac{3}{4}[1-u^2]du = \frac{3}{4}\left[u - \frac{u^3}{3}\right]_{-1}^{x-3} = \frac{3}{4}\left[x - \frac{7}{3} - \frac{(x-3)^3}{3}\right]$. Thus

$$F(x) = \begin{cases} 0 & x < 2 \\ \frac{1}{4}[3x - 7 - (x-3)^3] & 2 \le x \le 4 \\ 1 & x > 4 \end{cases}$$

b. By symmetry of f(x), $\tilde{m} = 3$

c. E(X) = $\int_{2}^{4} x \cdot \frac{3}{4}[1-(x-3)^2]dx = \frac{3}{4}\int_{-1}^{1}(y+3)(1-y^2)dx$

$$= \frac{3}{4}\left[3y + \frac{y^2}{2} - y^3 - \frac{y^4}{4}\right]_{-1}^{1} = \frac{3}{4} \cdot 4 = 3$$

V(X) = $\int_{-\infty}^{\infty}(x-m)^2 f(x)dx = \frac{3}{4}\int_{2}^{4}(x-3)^2 \cdot [1-(x-3)^2]dx$

$$= \frac{3}{4}\int_{-1}^{1} y^2(1-y^2)dy = \frac{3}{4} \cdot \frac{4}{15} = \frac{1}{5} = .2$$

18.

a. F(X) = $\frac{x-A}{B-A}$ = p $\Rightarrow$ x = (100p)th percentile = A + (B - A)p

b. $E(X) = \int_{A}^{B} x \cdot \frac{1}{B-A}dx = \frac{1}{B-A} \cdot \left.\frac{x^2}{2}\right]_{A}^{B} = \frac{1}{2} \cdot \frac{1}{B-A} \cdot \left(B^2 - A^2\right) = \frac{A+B}{2}$

$$E(X^2) = \frac{1}{3} \cdot \frac{1}{B-A} \cdot \left(B^3 - A^3\right) = \frac{A^2 + AB + B^2}{3}$$

$$V(X) = \left(\frac{A^2 + AB + B^2}{3}\right) - \left(\frac{(A+B)}{2}\right)^2 = \frac{(B-A)^2}{12}, \quad s_x = \frac{(B-A)}{\sqrt{12}}$$

c. $E(X^n) = \int_{A}^{B} x^n \cdot \frac{1}{B-A}dx = \frac{B^{n+1} - A^{n+1}}{(n+1)(B-A)}$

19.

a. $P(X \le 1) = F(1) = .25[1 + \ln(4)] \approx .597$

b. $P(1 \le X \le 3) = F(3) - F(1) \approx .966 - .597 \approx .369$

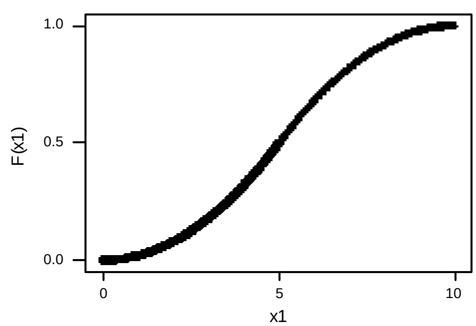
c. $f(x) = F'(x) = .25 \ln(4) - .25 \ln(x)$ for $o < x < 4$

20.

a. For $0 \le y \le 5$, F(y) = $\int_0^y \frac{1}{25} u\,du = \frac{y^2}{50}$

For $5 \le y \le 10$, F(y) = $\int_0^y f(u)du = \int_0^5 f(u)du + \int_5^y f(u)du$

$$= \frac{1}{2} + \int_0^y \left(\frac{2}{5} - \frac{u}{25}\right)du = \frac{2}{5}y - \frac{y^2}{50} - 1$$



b. For $0 < p \le .5$, p = F(y_p) = $\frac{y_p^2}{50} \Rightarrow y_p = (50p)^{1/2}$

For $.5 < p \le 1$, p = $\frac{2}{5}y_p - \frac{y_p^2}{50} - 1 \Rightarrow y_p = 10 - 5\sqrt{2(1-p)}$

c. E(Y) = 5 by straightforward integration (or by symmetry of f(y)), and similarly V(Y)= $\frac{50}{12} = 4.1667$. For the waiting time X for a single bus,

E(X) = 2.5 and V(X) = $\frac{25}{12}$

21. E(area) = E(πR^2) = $\int_{-\infty}^{\infty} pr^2 f(r)dr = \int_9^{11} pr^2 \left(\frac{3}{4}\right)\left(1 - (10-r)^2\right)dr$

$$= \left(\frac{3}{4}\right)p \int_9^{11} r^2\left(1 - (100 - 20r + r^2)\right)dr = \frac{3}{4}p \int_9^{11} -99r^2 + 20r^3 - r^4 dr = 100 \cdot 2p$$

22.

a. For $1 \le x \le 2$, F(x) = $\int_1^x 2\left(1-\frac{1}{y^2}\right)dy = 2\left(y+\frac{1}{y}\right)\Big]_1^x = 2\left(x+\frac{1}{x}\right)-4,$ so

$$F(x) = \begin{cases} 0 & x<1 \\ 2\left(x+\frac{1}{x}\right)-4 & 1\le x\le 2 \\ 1 & x>2 \end{cases}$$

b. $2\left(x_p+\frac{1}{x_p}\right)-4 = p \Rightarrow 2x_p^{\,2} - (4-p)x_p + 2 = 0 \Rightarrow x_p = \frac{1}{4}\left[4+p+\sqrt{p^2+8p}\right]$ To find $\tilde{m}$, set p = .5 $\Rightarrow$ $\tilde{m}$ = 1.64

c. E(X) = $\int_1^2 x\cdot 2\left(1-\frac{1}{x^2}\right)dx = 2\int_1^2\left(x-\frac{1}{x}\right)dx = 2\left(\frac{x^2}{2}-\ln(x)\right)\Big]_1^2 = 1.614$

E(X^2) = $2\int_1^2\left(x^2-1\right)dx = 2\left(\frac{x^3}{3}-x\right)\Big]_1^2 = \frac{8}{3} \Rightarrow$ Var(X) = .0626

d. Amount left = max(1.5 – X, 0), so

E(amount left) = $\int_1^2 \max(1.5-x,0)\,f(x)dx = 2\int_1^{1.5}(1.5-x)\left(1-\frac{1}{x^2}\right)dx = .061$

23. With X = temperature in °C, temperature in °F = $\frac{9}{5}X+32,$ so

$E\left[\frac{9}{5}X+32\right] = \frac{9}{5}(120)+32 = 248,\quad Var\left[\frac{9}{5}X+32\right] = \left(\frac{9}{5}\right)^2\cdot(2)^2 = 12.96,$
so $\sigma = 3.6$

24.

a. E(X) = $\int_q^\infty x \cdot \frac{kq^k}{x^{k+1}} dx = kq^k \int_q^\infty \frac{1}{x^k} dx = \frac{kq^k x^{-k+1}}{-k+1}\Bigg]_q^\infty = \frac{kq}{k-1}$

b. E(X) = ∞

c. E(X^2) = $kq^k \int_q^\infty \frac{1}{x^{k-1}} dx = \frac{kq^2}{k-2}$, so

Var(X) = $\left(\frac{kq^2}{k-2}\right) - \left(\frac{kq}{k-1}\right)^2 = \frac{kq^2}{(k-2)(k-1)^2}$

d. Var(x) = ∞, since E(X^2) = ∞.

e. E(X^n) = $kq^k \int_q^\infty x^{n-(k+1)} dx$, which will be finite if n – (k+1) < -1, i.e. if n<k.

25.

a. P(Y ≤ 1.8 $\tilde{m}$ + 32) = P(1.8X + 32 ≤ 1.8 $\tilde{m}$ + 32) = P(X ≤ $\tilde{m}$) = .5

b. 90^{th} for Y = 1.8η(.9) + 32 where η(.9) is the 90^{th} percentile for X, since
P(Y ≤ 1.8η(.9) + 32) = P(1.8X + 32 ≤ 1.8η(.9) + 32)
= (X ≤ η(.9)) = .9 as desired.

c. The (100p)th percentile for Y is 1.8η(p) + 32, verified by substituting p for .9 in the argument of **b**. When Y = aX + b, (i.e. a linear transformation of X), and the (100p)th percentile of the X distribution is η(p), then the corresponding (100p)th percentile of the Y distribution is a·η(p) + b. (same linear transformation applied to X's percentile)

Section 4.3

26.

a. $P(0 \le Z \le 2.17) = \Phi(2.17) - \Phi(0) = .4850$

b. $\Phi(1) - \Phi(0) = .3413$

c. $\Phi(0) - \Phi(-2.50) = .4938$

d. $\Phi(2.50) - \Phi(-2.50) = .9876$

e. $\Phi(1.37) = .9147$

f. $P(-1.75 < Z) + [1 – P(Z < -1.75)] = 1 - \Phi(-1.75) = .9599$

g. $\Phi(2) - \Phi(-1.50) = .9104$

h. $\Phi(2.50) - \Phi(1.37) = .0791$

i. $1 - \Phi(1.50) = .0668$

j. $P(|Z| \le 2.50) = P(-2.50 \le Z \le 2.50) = \Phi(2.50) - \Phi(-2.50) = .9876$

27.

a. .9838 is found in the 2.1 row and the .04 column of the standard normal table so c = 2.14.

b. $P(0 \le Z \le c) = .291 \Rightarrow \Phi(c) = .7910 \Rightarrow c = .81$

c. $P(c \le Z) = .121 \Rightarrow 1 - P(c \le Z) = P(Z < c) = \Phi(c) = 1 - .121 = .8790 \Rightarrow c = 1.17$

d. $P(-c \le Z \le c) = \Phi(c) - \Phi(-c) = \Phi(c) – (1 - \Phi(c)) = 2\Phi(c) – 1$
$\Rightarrow \Phi(c) = .9920 \Rightarrow c = .97$

e. $P(c \le |Z|) = .016 \Rightarrow 1 - .016 = .9840 = 1 – P(c \le |Z|) = P(|Z| < c)$
$= P(-c < Z < c) = \Phi(c) - \Phi(-c) = 2\Phi(c) – 1$
$\Rightarrow \Phi(c) = .9920 \Rightarrow c = 2.41$

28.

a. $\Phi(c) = .9100 \Rightarrow c \approx 1.34$ (.9099 is the entry in the 1.3 row, .04 column)

b. 9^{th} percentile = -91^{st} percentile = -1.34

c. $\Phi(c) = .7500 \Rightarrow c \approx .675$ since .7486 and .7517 are in the .67 and .68 entries, respectively.

d. $25^{th} = -75^{th} = -.675$

e. $\Phi(c) = .06 \Rightarrow c \approx .-1.555$ (both .0594 and .0606 appear as the –1.56 and –1.55 entries, respectively).

29.

a. Area under Z curve above $z_{.0055}$ is .0055, which implies that $\Phi(z_{.0055}) = 1 - .0055 = .9945$, so $z_{.0055} = 2.54$

b. $\Phi(z_{.09}) = .9100 \Rightarrow z = 1.34$ (since .9099 appears as the 1.34 entry).

c. $\Phi(z_{.633})$ = area below $z_{.633} = .3370 \Rightarrow z_{.633} \approx -.42$

30.

a. $P(X \le 100) = P\left(z \le \frac{100-80}{10}\right) = P(Z \le 2) = \Phi(2.00) = .9772$

b. $P(X \le 80) = P\left(z \le \frac{80-80}{10}\right) = P(Z \le 0) = \Phi(0.00) = .5$

c. $P(65 \le X \le 100) = P\left(\frac{65-80}{10} \le z \le \frac{100-80}{10}\right) = P(-1.50 \le Z \le 2)$
$= \Phi(2.00) - \Phi(-1.50) = .9772 - .0668 = .9104$

d. $P(70 \le X) = P(-1.00 \le Z) = 1 - \Phi(-1.00) = .8413$

e. $P(85 \le X \le 95) = P(.50 \le Z \le 1.50) = \Phi(1.50) - \Phi(.50) = .2417$

f. $P(|X - 80| \le 10) = P(-10 \le X - 80 \le 10) = P(70 \le X \le 90)$
$P(-1.00 \le Z \le 1.00) = .6826$

31.

a. $P(X \le 18) = P\left(z \le \frac{18-15}{1.25}\right) = P(Z \le 2.4) = \Phi(2.4) = .9452$

b. $P(10 \le X \le 12) = P(-4.00 \le Z \le -2.40) \approx P(Z \le -2.40) = \Phi(-2.40) = .0082$

c. $P(|X - 10| \le 2(1.25)) = P(-2.50 \le X\text{-}15 \le 2.50) = P(12.5 \le X \le 17.5)$
$P(-2.00 \le Z \le 2.00) = .9544$

32.

a. $P(X > .25) = P(Z > -.83) = 1 - .2033 = .7967$

b. $P(X \le .10) = \Phi(-3.33) = .0004$

c. We want the value of the distribution, c, that is the 95th percentile (5% of the values are higher). The 95th percentile of the standard normal distribution = 1.645. So c = .30 + (1.645)(.06) = .3987. The largest 5% of all concentration values are above .3987 mg/cm^3.

33.

a. $P(X \ge 10) = P(Z \ge .43) = 1 - \Phi(.43) = 1 - .6664 = .3336$.
$P(X > 10) = P(X \ge 10) = .3336$, since for any continuous distribution, $P(x = a) = 0$.

b. $P(X > 20) = P(Z > 4) \approx 0$

c. $P(5 \le X \le 10) = P(-1.36 \le Z \le .43) = \Phi(.43) - \Phi(-1.36) = .6664 - .0869 = .5795$

d. $P(8.8 - c \le X \le 8.8 + c) = .98$, so $8.8 - c$ and $8.8 + c$ are at the 1st and the 99th percentile of the given distribution, respectively. The 1st percentile of the standard normal distribution has the value –2.33, so
$8.8 - c = \mu + (-2.33)\sigma = 8.8 - 2.33(2.8) \Rightarrow c = 2.33(2.8) = 6.524$.

e. From a, $P(x > 10) = .3336$. Define event A as {diameter > 10}, then P(at least one A_i) =
$1 - P(\text{no } A_i) = 1 - P(A')^4 = 1 - (1 - .3336)^4 = 1 - .1972 = .8028$

34. Let X denote the diameter of a randomly selected cork made by the first machine, and let Y be defined analogously for the second machine.
$P(2.9 \le X \le 3.1) = P(-1.00 \le Z \le 1.00) = .6826$
$P(2.9 \le Y \le 3.1) = P(-7.00 \le Z \le 3.00) = .9987$
So the second machine wins handily.

35.

a. $\mu + \sigma \cdot$(91st percentile from std normal) = 30 + 5(1.34) = 36.7

b. 30 + 5(-1.555) = 22.225

c. μ = 3.000 μm; σ = 0.140. We desire the 90th percentile: 30 + 1.28(0.14) = 3.179

36. μ = 43; σ = 4.5

a. P(X < 40) = $P\left(z \le \frac{40-43}{4.5}\right)$ = P(Z < -0.667) = .2514

P(X > 60) = $P\left(z > \frac{60-43}{4.5}\right)$ = P(Z > 3.778) ≈ 0

b. 43 + (-0.67)(4.5) = 39.985

37. P(damage) = P(X < 100) = $P\left(z < \frac{100-200}{300}\right)$ = P(Z < -3.33) = .0004

P(at least one among five is damaged) = 1 – P(none damaged)
= 1 – (.9996)^5 = 1 - .998 = .002

38. From Table A.3, P(-1.96 ≤ Z ≤ 1.96) = .95. Then P(μ - .1 ≤ X ≤ μ + .1) =
$P\left(\frac{-.1}{s} < z < \frac{.1}{s}\right)$ implies that $\frac{.1}{s}$ = 1.96, and thus that $s = \frac{.1}{1.96} = .0510$

39. Since 1.28 is the 90th z percentile ($z_{.1}$ = 1.28) and –1.645 is the 5th z percentile ($z_{.05}$ = 1.645), the given information implies that $\mu + \sigma(1.28) = 10.256$ and $\mu + \sigma(-1.645) = 9.671$, from which $\sigma(-2.925) = -.585$, σ = .2000, and μ = 10.

40.

a. P(μ - 1.5σ ≤ X ≤ μ + 1.5σ) = P(-1.5 ≤ Z ≤ 1.5) = Φ(1.50) - Φ(-1.50) = .8664

b. P(X < μ - 2.5σ or X > μ + 2.5σ) = 1 - P(μ - 2.5σ ≤ X ≤ μ + 2.5σ)
= 1 - P(-2.5 ≤ Z ≤ 2.5) = 1 - .9876 = .0124

c. P(μ - 2σ ≤ X ≤ μ - σ or μ + σ ≤ X ≤ μ + 2σ) = P(within 2 sd's) – P(within 1 sd) = P(μ - 2σ ≤ X ≤ μ + 2σ) - P(μ - σ ≤ X ≤ μ + σ)
= .9544 - .6826 = .2718

41. With μ = .500 inches, the acceptable range for the diameter is between .496 and .504 inches, so unacceptable bearings will have diameters smaller than .496 or larger than .504. The new distribution has μ = .499 and σ =.002. P(x < .496 or x >.504) =

$$P\left(z < \frac{.496-.499}{.002}\right) + P\left(z > \frac{.504-.499}{.002}\right) = P(z < -1.5) + P(z > 2.5)$$

$\Phi(-1.5) + (1-\Phi(2.5)) = .0068 + .0062 = .073$, or 7.3% of the bearings will be unacceptable.

42.

a. $P(67 \le X \le 75) = P(-1.00 \le Z \le 1.67) = .7938$

b. $P(70 - c \le X \le 70 + c) = P\left(\frac{-c}{3} \le Z \le \frac{c}{3}\right) = 2\Phi(\frac{c}{3}) - 1 = .95 \Rightarrow \Phi(\frac{c}{3}) = .9750$

$$\frac{c}{3} = 1.96 \Rightarrow c = 5.88$$

c. 10·P(a single one is acceptable) = 9.05

d. p = P(X < 73.84) = P(Z < 1.28) = .9, so P(Y ≤ 8) = B(8;10,.9) = .264

43. The stated condition implies that 99% of the area under the normal curve with μ = 10 and σ = 2 is to the left of c – 1, so c – 1 is the 99th percentile of the distribution. Thus c – 1 = μ + σ(2.33) = 20.155, and c = 21.155.

44.

a. By symmetry, $P(-1.72 \le Z \le -.55) = P(.55 \le Z \le 1.72) = \Phi(1.72) - \Phi(.55)$

b. $P(-1.72 \le Z \le .55) = \Phi(.55) - \Phi(-1.72) = \Phi(.55) - [1 - \Phi(1.72)]$
No, symmetry of the Z curve about 0.

45. X ~N(3432, 482)

a. $$P(x > 4000) = P\left(Z > \frac{4000-3432}{482}\right) = P(z > 1.18)$$
$$= 1 - \Phi(1.18) = 1 - .8810 = .1190$$
$$P(3000 < x < 4000) = P\left(\frac{3000-3432}{482} < Z < \frac{4000-3432}{482}\right)$$
$$= \Phi(1.18) - \Phi(-.90) = .8810 - .1841 = .6969$$

b. $$P(x < 2000 or x > 5000) = P\left(Z < \frac{2000-3432}{482}\right) + P\left(Z > \frac{5000-3432}{482}\right)$$
$$= \Phi(-2.97) + [1 - \Phi(3.25)] = .0015 + .0006 = .0021$$

c. We will use the conversion 1 lb = 454 g, then 7 lbs = 3178 grams, and we wish to find

$$P(x > 3178) = P\left(Z > \frac{3178 - 3432}{482}\right) = 1 - \Phi(-.53) = .7019$$

d. We need the top .0005 and the bottom .0005 of the distribution. Using the Z table, both .9995 and .0005 have multiple z values, so we will use a middle value, ±3.295. Then 3432±(482)3.295 = 1844 and 5020, or the most extreme .1% of all birth weights are less than 1844 g and more than 5020 g.

e. Converting to lbs yields mean 7.5595 and s.d. 1.0608. Then

$P(x > 7) = P\left(Z > \frac{7 - 7.5595}{1.0608}\right) = 1 - \Phi(-.53) = .7019$ This yields the same answer as in part c.

46. We use a Normal approximation to the Binomial distribution: $X \sim b(x;1000,.03) \,\tilde{}\, N(30,5.394)$

a. $$P(x \ge 40) = 1 - P(x \le 39) = 1 - P\left(Z \le \frac{39.5 - 30}{5.394}\right)$$
$$= 1 - \Phi(1.76) = 1 - .9608 = .0392$$

b. 5% of 1000 = 50: $P(x \le 50) = P\left(Z \le \frac{50.5 - 30}{5.394}\right) = \Phi(3.80) \approx 1.00$

47. $P(|X - \mu| \ge \sigma) = P(X \le \mu - \sigma \text{ or } X \ge \mu + \sigma)$
$= 1 - P(\mu - \sigma \le X \le \mu + \sigma) = 1 - P(-1 \le Z \le 1) = .3174$
Similarly, $P(|X - \mu| \ge 2\sigma) = 1 - P(-2 \le Z \le 2) = .0456$
And $P(|X - \mu| \ge 3\sigma) = 1 - P(-3 \le Z \le 3) = .0026$

48.

a. $P(20 - .5 \le X \le 30 + .5) = P(19.5 \le X \le 30.5) = P(-1.1 \le Z \le 1.1) = .7286$

b. P(at most 30) = $P(X \le 30 + .5) = P(Z \le 1.1) = .8643$.
P(less than 30) = $P(X < 30 - .5) = P(Z < .9) = .8159$

49.

P:	.5	.6	.8
μ:	12.5	15	20
σ:	2.50	2.45	2.00

a.

	P(15≤ X ≤20)	P(14.5 ≤ normal ≤ 20.5)
.5	.212	$P(.80 \le Z \le 3.20) = .2112$
.6	.577	$P(-.20 \le Z \le 2.24) = .5668$
.8	.573	$P(-2.75 \le Z \le .25) = .5957$

b.

P(X ≤15)	P(normal ≤ 15.5)
.885	$P(Z \le 1.20) = .8849$
.575	$P(Z \le .20) = .5793$
.017	$P(Z \le -2.25) = .0122$

c.

P(20 ≤X)	P(19.5 ≤ normal)
.002	.0026
.029	.0329
.617	.5987

50. P = .10; n = 200; np = 20, npq = 18

a. $P(X \le 30) = \Phi\left(\frac{30+.5-20}{\sqrt{18}}\right) = \Phi(2.47) = .9932$

b. $P(X < 30) = P(X \le 29) = \Phi\left(\frac{29+.5-20}{\sqrt{18}}\right) = \Phi(2.24) = .9875$

c. $P(15 \le X \le 25) = P(X \le 25) - P(X \le 14) = \Phi\left(\frac{25+.5-20}{\sqrt{18}}\right) - \Phi\left(\frac{14+.5-20}{\sqrt{18}}\right)$

$\Phi(1.30) - \Phi(-1.30) = .9032 - .0968 = .8064$

51. N = 500, p = .4, $\mu = 200$, $\sigma = 10.9545$

a. $P(180 \le X \le 230) = P(179.5 \le \text{normal} \le 230.5) = P(-1.87 \le Z \le 2.78) = .9666$

b. $P(X < 175) = P(X \le 174) = P(\text{normal} \le 174.5) = P(Z \le -2.33) = .0099$

52. P(X ≤ μ + σ[(100p)th percentile for std normal])

$P\left(\frac{X-m}{s} \le [...]\right)$ = P(Z ≤ […]) = p as desired

53.

a. $F_y(y) = P(Y \le y) = P(aX + b \le y) = P\left(X \le \frac{(y-b)}{a}\right)$ (for a > 0).

Now differentiate with respect to y to obtain

$f_y(y) = F_y'(y) = \frac{1}{\sqrt{2p}\, a s} e^{-\frac{1}{2a^2 s^2}[y-(am+b)]^2}$ so Y is normal with mean aμ + b and variance $a^2\sigma^2$.

b. Normal, mean $\frac{9}{5}(115) + 32 = 239$, variance = 12.96

54.

a. $P(Z \ge 1) \approx .5 \cdot \exp\left(\frac{83 + 351 + 562}{703 + 165}\right) = .1587$

b. $P(Z > 3) \approx .5 \cdot \exp\left(\frac{-2362}{399.3333}\right) = .0013$

c. $P(Z > 4) \approx .5 \cdot \exp\left(\frac{-3294}{340.75}\right) = .0000317$, so

$P(-4 < Z < 4) \approx 1 - 2(.0000317) = .999937$

d. $P(Z > 5) \approx .5 \cdot \exp\left(\frac{-4392}{305.6}\right) = .00000029$

Section 4.4

55.

a. $\Gamma(6) = 5! = 120$

b. $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \left(\frac{3}{4}\right)\sqrt{p} \approx 1.329$

c. $F(4;5) = .371$ from row 4, column 5 of Table A.4

d. $F(5;4) = .735$

e. $F(0;4) = P(X \le 0; \alpha = 4) = 0$

56.

a. $P(X \le 5) = F(5;7) = .238$

b. $P(X < 5) = P(X \le 5) = .238$

c. $P(X > 8) = 1 - P(X < 8) = 1 - F(8;7) = .313$

d. $P(3 \le X \le 8) = F(8;7) - F(3;7) = .653$

e. $P(3 < X < 8) =.653$

f. $P(X < 4 \text{ or } X > 6) = 1 - P(4 \le X \le 6) = 1 - [F(6;7) - F(4;7)] = .713$

57.

a. $\mu = 20,\ \sigma^2 = 80 \Rightarrow \alpha\beta = 20, \alpha\beta^2 = 80 \Rightarrow \beta = \frac{80}{20}, \alpha = 5$

b. $P(X \le 24) = F\left(\frac{24}{4};5\right) = F(6;5) = .715$

c. $P(20 \le X \le 40) = F(10;5) - F(5;5) = .411$

58. $\mu = 24,\ \sigma^2 = 144 \Rightarrow \alpha\beta = 24, \alpha\beta^2 = 144 \Rightarrow \beta = 6, \alpha = 4$

a. $P(12 \le X \le 24) = F(4;4) - F(2;4) = .424$

b. $P(X \le 24) = F(4;4) = .567$, so while the mean is 24, the median is less than 24. ($P(X \le \tilde{m}) = .5$); This is a result of the positive skew of the gamma distribution.

c. We want a value of X for which F(X;4)=.99. In table A.4, we see F(10;4)=.990. So with $\beta = 6$, the 99th percentile = 6(10)=60.

d. We want a value of X for which F(X;4)=.995. In the table, F(11;4)=.995, so t = 6(11)=66. At 66 weeks, only .5% of all transistors would still be operating.

59.

a. $E(X) = \frac{1}{l} = 1$

b. $s = \frac{1}{l} = 1$

c. $P(X \le 4) = 1 - e^{-(1)(4)} = 1 - e^{-4} = .982$

d. $P(2 \le X \le 5) = 1 - e^{-(1)(5)} - \left[1 - e^{-(1)(2)}\right] = e^{-2} - e^{-5} = .129$

60.

a. $P(X \le 100) = 1 - e^{-(100)(.01386)} = 1 - e^{-1.386} = .7499$

$P(X \le 200) = 1 - e^{-(200)(.01386)} = 1 - e^{-2.772} = .9375$

$P(100 \le X \le 200) = P(X \le 200) - P(X \le 100) = .9375 - .7499 = .1876$

b. $\mu = \frac{1}{.01386} = 72.15$, $\sigma = 72.15$

$P(X > \mu + 2\sigma) = P(X > 72.15 + 2(72.15)) = P(X > 216.45) =$

$1 - \left[1 - e^{-(216.45)(.01386)}\right] = e^{-2.9999} = .0498$

c. $.5 = P(X \le \tilde{m}) \Rightarrow 1 - e^{-(\tilde{m})(.01386)} = .5 \Rightarrow e^{-(\tilde{m})(.01386)} = .5$

$-\tilde{m}(.01386) = \ln(.5) = .693 \Rightarrow \tilde{m} = 50$

61. Mean $= \frac{1}{l} = 25{,}000$ implies $\lambda = .00004$

a. $P(X > 20{,}000) = 1 - P(X \le 20{,}000) = 1 - F(20{,}000; .00004) = e^{-(.00004)(20{,}000)} = .449$

$P(X \le 30{,}000) = F(30{,}000; .00004) = e^{-1.2} = .699$

$P(20{,}000 \le X \le 30{,}000) = .699 - .551 = .148$

b. $s = \frac{1}{l} = 25{,}000$, so $P(X > \mu + 2\sigma) = P(x > 75{,}000) =$

$1 - F(75{,}000; .00004) = .05$.

Similarly, $P(X > \mu + 3\sigma) = P(x > 100{,}000) = .018$

62.

a. E(X) = αβ = $n\frac{1}{\lambda} = \frac{n}{\lambda}$; for λ = .5, n = 10, E(X) = 20

b. P(X ≤ 30) = $F\left(\frac{30}{2};10\right)$ = F(15;10) = .930

c. P(X ≤ t) = P(at least n events in time t) = P(Y ≥ n) when Y ~ Poisson with parameter λt .
Thus P(X ≤ t) = 1 – P(Y < n) = 1 – P(Y ≤ n – 1) $= 1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$.

63.

a. {X ≥ t} = $A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$

b. P(X ≥ t) =P(A_1) · P(A_2) · P(A_3) · P(A_4) · P(A_5) = $\left(e^{-\lambda t}\right)^5 = e^{-.05t}$, so $F_x(t)$ = P(X ≤ t) = 1 - $e^{-.05t}$, $f_x(t)$ = $.05e^{-.05t}$ for t ≥ 0. Thus X also ha an exponential distribution , but with parameter λ = .05.

c. By the same reasoning, P(X ≤ t) = 1 - $e^{-n\lambda t}$, so X has an exponential distribution with parameter nλ.

64. With x_p = (100p)th percentile, p = F(x_p) = 1 - $e^{-\lambda x_p} \Rightarrow e^{-\lambda x_p} = 1 - p$,
$\Rightarrow -\lambda x_p = \ln(1-p) \Rightarrow x_p = \frac{-[\ln(1-p)]}{\lambda}$. For p = .5, $x_{.5}$ = $\tilde{\mu} = \frac{.693}{\lambda}$.

65.

a. {X^2 ≤ y} = $\left\{-\sqrt{y} \le X \le \sqrt{y}\right\}$

b. P(X^2 ≤ y) = $\int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$. Now differentiate with respect to y to obtain the chi-squared p.d.f. with ν = 1.

Section 4.5

66.

a. E(X) = $3\Gamma\left(1+\frac{1}{2}\right) = 3\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = 2.66$,

Var(X) = $9\left[\Gamma(1+1) - \Gamma^2\left(1+\frac{1}{2}\right)\right] = 1.926$

b. P(X ≤ 6) = $1 - e^{-(6/b)^a} = 1 - e^{-(6/3)^2} = 1 - e^{-4} = .982$

c. P(1.5 ≤ X ≤ 6) = $1 - e^{-(6/3)^2} - \left[1 - e^{-(1.5/3)^2}\right] = e^{-.25} - e^{-4} = .760$

67.

a. P(X ≤ 250) = F(250;2.5, 200) = $1 - e^{-(250/200)^{2.5}} = 1 - e^{-1.75} \approx .8257$
P(X < 250) = P(X ≤ 250) ≈ .8257
P(X > 300) = 1 – F(300; 2.5, 200) = $e^{-(1.5)^{2.5}} = .0636$

b. P(100 ≤ X ≤ 250) = F(250;2.5, 200) - F(100;2.5, 200) ≈ .8257 - .162 = .6637

c. The median $\tilde{m}$ is requested. The equation F($\tilde{m}$) = .5 reduces to

.5 = $e^{-(\tilde{m}/200)^{2.5}}$, i.e., ln(.5) ≈ $-\left(\frac{\tilde{m}}{200}\right)^{2.5}$, so $\tilde{m}$ = $(.6931)^{.4}(200)$ = 172.727.

68.

a. For x > 3.5, F(x) = P(X ≤ x) = P(X – 3.5 ≤ x – 3.5) = 1 - $e^{-\left[\frac{(x-3.5)}{1.5}\right]^2}$

b. E(X – 3.5) = $1.5\Gamma\left(\frac{3}{2}\right)$ = 1.329 so E(X) = 4.829

Var(X) = Var(X – 3.5) = $(1.5)^2\left[\Gamma(2) - \Gamma^2\left(\frac{3}{2}\right)\right] = .483$

c. P(X > 5) = 1 – P(X ≤ 5) = $1 - \left[1 - e^{-1}\right] = e^{-1} = .368$

d. P(5 ≤ X ≤ 8) = $1 - e^{-9} - \left[1 - e^{-1}\right] = e^{-1} - e^{-9} = .3679 - .0001 = .3678$

69. $\mu = \int_0^\infty x \cdot \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-(x/\beta)^\alpha} dx$ = (after y = $\left(\frac{x}{\beta}\right)^\alpha$, dy = $\frac{\alpha x^{\alpha-1}}{\beta^\alpha}$ dx)

$\beta \int_0^\infty y^{1/\alpha} e^{-y} dy = \beta \cdot \Gamma\left(1+\frac{1}{\alpha}\right)$ by definition of the gamma function.

70.

a. $.5 = F(\tilde{\mu}) = 1 - e^{-(\tilde{\mu}/3)^2} \Rightarrow$

$e^{-\tilde{\mu}^2/9} = .5 \Rightarrow \tilde{\mu}^2 = -9\ln(.5) = 6.2383 \Rightarrow \tilde{\mu} = 2.50$

b. $1 - e^{-[(\tilde{\mu}-3.5)/1.5]^2} = .5 \Rightarrow (\tilde{\mu}-3.5)^2$ = -2.25 ln(.5) = 1.5596 $\Rightarrow \tilde{\mu}$ = 4.75

c. P = F(x_p) = 1 - $e^{-(x_p/\beta)^\alpha}$ $\Rightarrow$ $(x_p/\beta)^\alpha$ = -ln(1 – p) $\Rightarrow$ x_p = β[-ln(1-p)]$^{1/\alpha}$

d. The desired value of t is the 90th percentile (since 90% will not be refused and 10% will be). From **c,** the 90th percentile of the distribution of X – 3.5 is 1.5[-ln(.1)]$^{1/2}$ = 2.27661, so t = 3.5 + 2.2761 = 5.7761

71. X ~ Weibull: α=20,β=100

a. $F(x,20,\beta) = 1 - e^{-\left(\frac{x}{\beta}\right)^\alpha} = 1 - e^{-\left(\frac{105}{100}\right)^{20}} = 1 - .070 = .930$

b. $F(105) - F(100) = .930 - \left(1 - e^{-1}\right) = .930 - .632 = .298$

c. $.50 = 1 - e^{-\left(\frac{x}{100}\right)^{20}} \Rightarrow e^{-\left(\frac{x}{100}\right)^{20}} = .50 \Rightarrow -\left(\frac{x}{100}\right)^{20} = \ln(.50)$

$\left(\frac{-x}{100}\right) = \sqrt[20]{\ln(.50)} \Rightarrow -x = 100\left(\sqrt[20]{\ln(.50)}\right) \Rightarrow x = 98.18$

72.

a. $E(X) = e^{\left(\mu + \frac{\sigma^2}{2}\right)} = e^{4.82} = 123.97$

$V(X) = \left(e^{\left(2(4.5)+.8^2\right)}\right) \cdot \left(e^{.8} - 1\right) = (15{,}367.34)(.8964) = 13{,}776.53$

$\sigma = 117.373$

b. $P(x \le 100) = P\left(z \le \frac{\ln(100) - 4.5}{.8}\right) = \Phi(0.13) = .5517$

c. $P(x \ge 200) = P\left(z \ge \frac{\ln(200) - 4.5}{.8}\right) = 1 - \Phi(1.00) = 1 - .8413 = .1587 = P(x > 200)$

73.

a. E(X) = $e^{3.5+(1.2)^2/2}$ = 68.0335; V(X) = $e^{2(3.5)+(1.2)^2} \cdot \left(e^{(1.2)^2} - 1\right) = 14907.168$;
σ_x = 122.0949

b. P(50 ≤ X ≤ 250) = $P\left(z \le \dfrac{\ln(250) - 3.5}{1.2}\right) - P\left(z \le \dfrac{\ln(50) - 3.5}{1.2}\right)$
P(Z ≤ 1.68) – P(Z ≤ .34) = .9535 - .6331 = .3204.

c. P(X ≤ 68.0335) = $P\left(z \le \dfrac{\ln(68.0335) - 3.5}{1.2}\right)$ = P(Z ≤ .60) = .7257. The lognormal distribution is not a symmetric distribution.

74.

a. .5 = F($\tilde{m}$) = $\Phi\left(\dfrac{\ln(\tilde{m}) - m}{s}\right)$, (where $\tilde{m}$ refers to the lognormal distribution and μ and σ to the normal distribution). Since the median of the standard normal distribution is 0,
$\dfrac{\ln(\tilde{m}) - m}{s} = 0$, so ln($\tilde{m}$) = μ ⇒ $\tilde{m} = e^{m}$. For the power distribution,
$\tilde{m} = e^{3.5} = 33.12$

b. 1 - α = Φ(z_α) = P(Z ≤ z_α) = $\left(\dfrac{\ln(X) - m}{s} \le z_a\right) = P(\ln(X) \le m + sz_a)$
= $P(X \le e^{m+sz_a})$, so the 100(1 - α)th percentile is e^{m+sz_a} . For the power distribution, the 95th percentile is $e^{3.5+(1.645)(1.2)} = e^{5.474} = 238.41$

75.

a. E(X) = $e^{5+(.01)/2} = e^{5.005} = 149.157$; Var(X) = $e^{10+(.01)} \cdot \left(e^{.01} - 1\right) = 223.594$

b. P(X > 125) = 1 – P(X ≤ 125) =
$$= 1 - P\left(z \le \frac{\ln(125) - 5}{.1}\right) = 1 - \Phi(-1.72) = .9573$$

c. P(110 ≤ X ≤ 125) $= \Phi(-1.72) - \Phi\left(\dfrac{\ln(110) - 5}{.1}\right) = .0427 - .0013 = .0414$

d. $\tilde{m} = e^{5} = 148.41$ (continued)

e. P(any particular one has X > 125) = .9573 ⇒ expected # = 10(.9573) = 9.573

f. We wish the 5th percentile, which is $e^{5+(-1.645)(.1)} = 125.90$

76.

a. E(X) = $e^{1.9+9^2/2} = 10.024$; Var(X) = $e^{3.8+(.81)} \cdot \left(e^{.81} - 1\right) = 125.395$, $\sigma_x = 11.20$

b. P(X ≤ 10) = P(ln(X) ≤ 2.3026) = P(Z ≤ .45) = .6736
P(5 ≤ X ≤ 10) = P(1.6094 ≤ ln(X) ≤2.3026)
= P(-.32 ≤ Z ≤ .45) = .6736 - .3745 = .2991

77. The point of symmetry must be $\frac{1}{2}$, so we require that $f\left(\frac{1}{2} - m\right) = f\left(\frac{1}{2} + m\right)$, i.e.,
$\left(\frac{1}{2} - m\right)^{a-1}\left(\frac{1}{2} + m\right)^{b-1} = \left(\frac{1}{2} + m\right)^{a-1}\left(\frac{1}{2} - m\right)^{b-1}$, which in turn implies that $\alpha = \beta$.

78.

a. E(X) = $\frac{5}{(5+2)} = \frac{5}{7} = .714$, V(X) = $\frac{10}{(49)(8)} = .0255$

b. f(x) = $\frac{\Gamma(7)}{\Gamma(5)\Gamma(2)} \cdot x^4 \cdot (1-x) = 30\left(x^4 - x^5\right)$ for 0 ≤ X ≤ 1,

so P(X ≤ .2) = $\int_0^{.2} 30\left(x^4 - x^5\right)dx = .0016$

c. P(.2 ≤ X ≤ .4) = $\int_{.2}^{.4} 30\left(x^4 - x^5\right)dx = .03936$

d. E(1 – X) = 1 – E(X) = 1 - $\frac{5}{7} = \frac{2}{7} = .286$

79.

a. E(X) = $\int_0^1 x \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^a (1-x)^{b-1} dx$

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)} = \frac{a\Gamma(a)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(a+b)}{(a+b)\Gamma(a+b)} = \frac{a}{a+b}$$

b. E[(1 – X)^m] = $\int_0^1 (1-x)^m \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}dx$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^{a-1}(1-x)^{m+b-1}dx = \frac{\Gamma(a+b)\cdot\Gamma(m+b)}{\Gamma(a+b+m)\Gamma(b)}$$

For m = 1, E(1 – X) = $\frac{b}{a+b}$.

80.

a. E(Y) = 10 $\Rightarrow E\left(\frac{Y}{20}\right) = \frac{1}{2} = \frac{a}{a+b}$; Var(Y) = $\frac{100}{7} \Rightarrow Var\left(\frac{Y}{20}\right) = \frac{100}{2800} = \frac{1}{28}$

$\frac{ab}{(a+b)^2(a+b+1)} \Rightarrow a = 3,\ b = 3$, after some algebra.

b. P(8 ≤ X ≤ 12) = $F\left(\frac{12}{20};3,3\right) - F\left(\frac{8}{20};3,3\right)$ = F(.6;3,3) – F(.4; 3,3).

The standard density function here is $30y^2(1-y)^2$,

so P(8 ≤ X ≤ 12) = $\int_{.4}^{.6} 30y^2(1-y)^2\,dy = .365$.

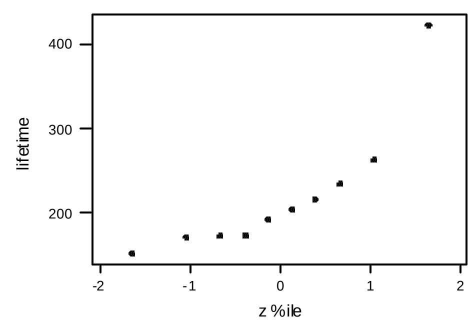
c. We expect it to snap at 10, so P(Y < 8 or Y > 12) = 1 - P(8 ≤ X ≤ 12)
= 1 - .365 = .665.

Section 4.6

81. The given probability plot is quite linear, and thus it is quite plausible that the tension distribution is normal.

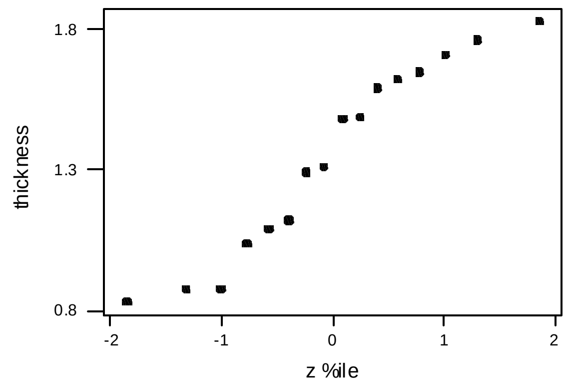
82. The z percentiles and observations are as follows:

percentile	observation
-1.645	152.7
-1.040	172.0
-0.670	172.5
-0.390	173.3
-0.130	193.0
0.130	204.7
0.390	216.5
0.670	234.9
1.040	262.6
1.645	422.6

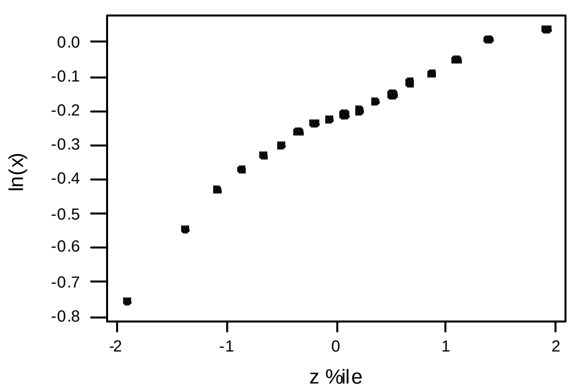


The accompanying plot is quite straight except for the point corresponding to the largest observation. This observation is clearly much larger than what would be expected in a normal random sample. Because of this outlier, it would be inadvisable to analyze the data using any inferential method that depended on assuming a normal population distribution.

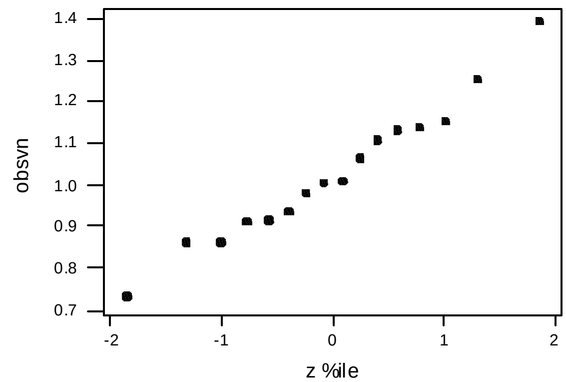
83. The z percentile values are as follows: -1.86, -1.32, -1.01, -0.78, -0.58, -0.40, -0.24,-0.08, 0.08, 0.24, 0.40, 0.58, 0.78, 1.01, 1.30, and 1.86. The accompanying probability plot is reasonably straight, and thus it would be reasonable to use estimating methods that assume a normal population distribution.



84. The Weibull plot uses ln(observations) and the z percentiles of the p_i values given. The accompanying probability plot appears sufficiently straight to lead us to agree with the argument that the distribution of fracture toughness in concrete specimens could well be modeled by a Weibull distribution.

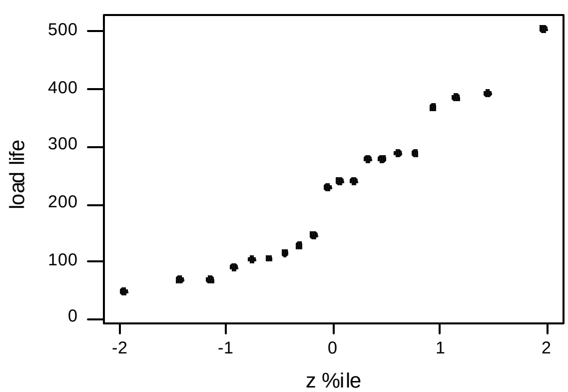


85. The (z percentile, observation) pairs are (-1.66, .736), (-1.32, .863), (-1.01, .865), (-.78, .913), (-.58, .915), (-.40, .937), (-.24, .983), (-.08, 1.007), (.08, 1.011), (.24, 1.064), (.40, 1.109), (.58, 1.132), (.78, 1.140), (1.01, 1.153), (1.32, 1.253), (1.86, 1.394). The accompanying probability plot is very straight, suggesting that an assumption of population normality is extremely plausible.

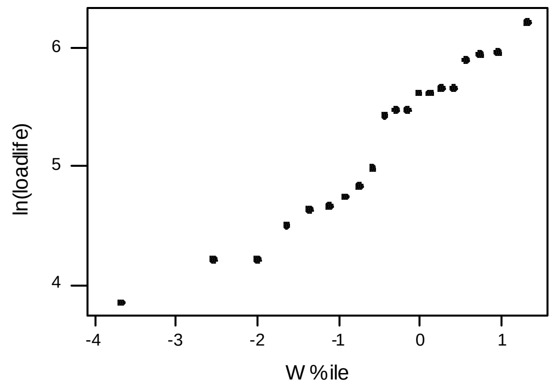


86.

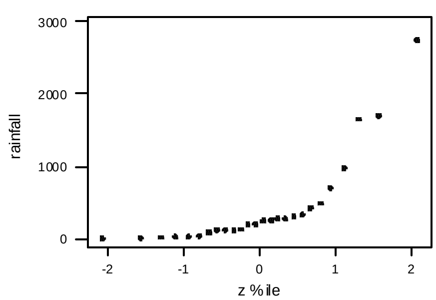
a. The 10 largest z percentiles are 1.96, 1.44, 1.15, .93, .76, .60, .45, .32, .19 and .06; the remaining 10 are the negatives of these values. The accompanying normal probability plot is reasonably straight. An assumption of population distribution normality is plausible.

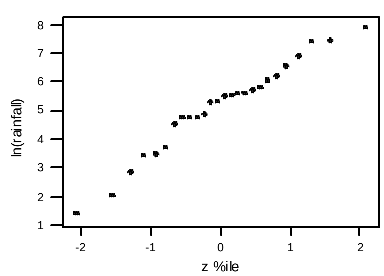


b. For a Weibull probability plot, the natural logs of the observations are plotted against extreme value percentiles; these percentiles are -3.68, -2.55, -2.01, -1.65, -1.37, -1.13, -.93, -.76, -.59, -.44, -.30, -.16, -.02, .12, .26, .40, .56, .73, .95, and 1.31. The accompanying probability plot is roughly as straight as the one for checking normality (a plot of ln(x) versus the z percentiles, appropriate for checking the plausibility of a lognormal distribution, is also reasonably straight - any of 3 different families of population distributions seems plausible.)



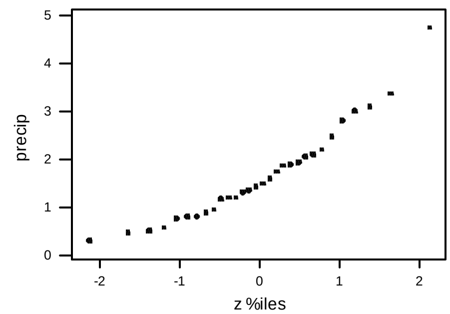
87. To check for plausibility of a lognormal population distribution for the rainfall data of Exercise 81 in Chapter 1, take the natural logs and construct a normal probability plot. This plot and a normal probability plot for the original data appear below. Clearly the log transformation gives quite a straight plot, so lognormality is plausible. The curvature in the plot for the original data implies a positively skewed population distribution - like the lognormal distribution.



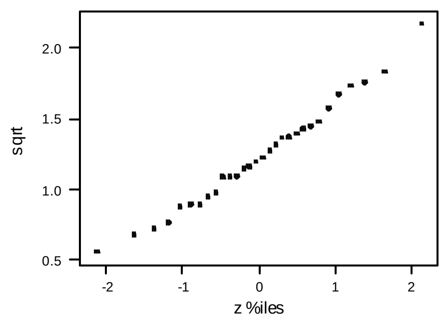


88.

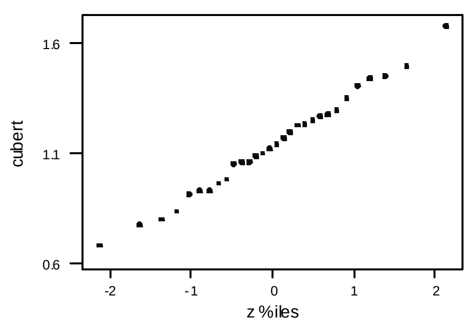
a. The plot of the original (untransformed) data appears somewhat curved.



b. The square root transformation results in a very straight plot. It is reasonable that this distribution is normally distributed.

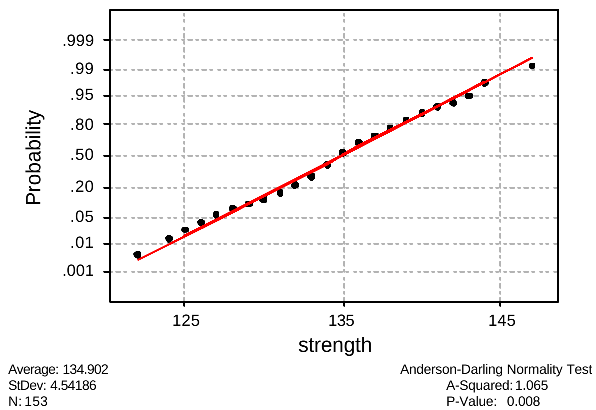


c. The cube root transformation also results in a very straight plot. It is very reasonable that the distribution is normally distributed.



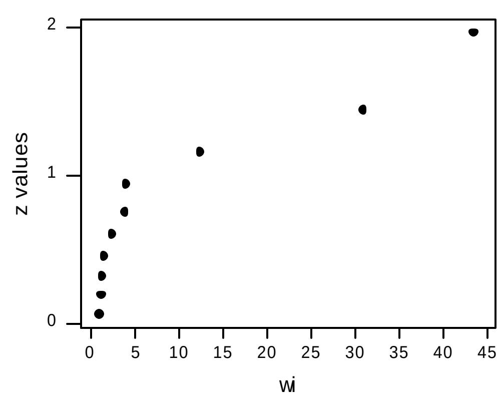
89. The pattern in the plot (below, generated by Minitab) is quite linear. It is very plausible that strength is normally distributed.

Normal Probability Plot



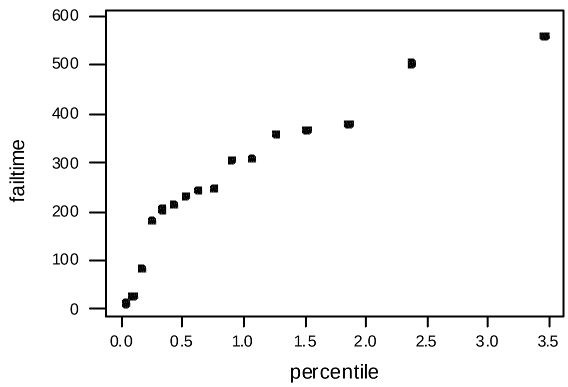
90. We use the data (table below) to create the desired plot.

ordered absolute values (w's)	probabilities	z values
0.89	0.525	0.063
1.15	0.575	0.19
1.27	0.625	0.32
1.44	0.675	0.454
2.34	0.725	0.6
3.78	0.775	0.755
3.96	0.825	0.935
12.38	0.875	1.15
30.84	0.925	1.44
43.4	0.975	1.96



This half-normal plot reveals some extreme values, without which the distribution may appear to be normal.

91. The (100p)th percentile $\eta(p)$ for the exponential distribution with $\lambda = 1$ satisfies $F(\eta(p)) = 1 - \exp[-\eta(p)] = p$, i.e., $\eta(p) = -\ln(1 - p)$. With n = 16, we need $\eta(p)$ for $p = \frac{5}{16}, \frac{1.5}{16}, \ldots, \frac{15.5}{16}$. These are .032, .398, .170, .247, .330, .421, .521, .633, .758, .901, 1.068, 1.269, 1.520, 1.856, 2.367, 3.466. this plot exhibits substantial curvature, casting doubt on the assumption of an exponential population distribution. Because λ is a scale parameter (as is σ for the normal family), $\lambda = 1$ can be used to assess the plausibility of the entire exponential family.



Supplementary Exercises

92.

a. $P(10 \le X \le 20) = \dfrac{10}{25} = .4$

b. $P(X \ge 10) = P(10 \le X \le 25) = \dfrac{15}{25} = .6$

c. For $0 \le X \le 25$, $F(x) = \int_0^x \dfrac{1}{25}\,dy = \dfrac{x}{25}$. F(x)=0 for x < 0 and = 1 for x > 25.

d. $E(X) = \dfrac{(A+B)}{2} = \dfrac{(0+25)}{2} = 12.5$; $Var(X) = \dfrac{(B-A)^2}{12} = \dfrac{625}{12} = 52.083$

93.

a. For $0 \le Y \le 25$, F(y) = $\frac{1}{24}\int_0^y \left(u - \frac{u^2}{12}\right) = \frac{1}{24}\left(\frac{u^2}{2} - \frac{u^3}{36}\right)\Bigg]_0^y$. Thus

F(y) = $\begin{cases} 0 & y < 0 \\ \frac{1}{48}\left(y^2 - \frac{y^3}{18}\right) & 0 \le y \le 12 \\ 1 & y > 12 \end{cases}$

b. P(Y ≤ 4) = F(4) = .259, P(Y > 6) = 1 – F(6) = .5
P(4 ≤ X ≤ 6) = F(6) – F(4) = .5 - .259 = .241

c. E(Y) = $\frac{1}{24}\int_0^{12} y^2\left(1 - \frac{y}{12}\right)dy = \frac{1}{24}\left[\frac{y^3}{3} - \frac{y^4}{48}\right]_0^{12} = 6$

E(Y^2) = $\frac{1}{24}\int_0^{12} y^3\left(1 - \frac{y}{12}\right)dy = 43.2$, so V(Y) = 43.2 – 36 = 7.2

d. P(Y < 4 or Y > 8) = 1 - P(4 ≤ X ≤ 8) = .518

e. the shorter segment has length min(Y, 12 – Y) so

E[min(Y, 12 – Y)] = $\int_0^{12} \min(y, 12-y)\cdot f(y)dy = \int_0^6 \min(y,12-y)\cdot f(y)dy$

$+\int_6^{12} \min(y,12-y)\cdot f(y)dy = \int_0^6 y\cdot f(y)dy + \int_6^{12}(12-y)\cdot f(y)dy = \frac{90}{24} = .3.75$

94.

a. Clearly f(x) ≥ 0. The c.d.f. is , for x > 0,

$$F(x) = \int_{-\infty}^x f(y)dy = \int_0^x \frac{32}{(y+4)^3}dy = -\frac{1}{2}\cdot\frac{32}{(y+4)^2}\Bigg]_0^x = 1 - \frac{16}{(x+4)^2}$$

(F(x) = 0 for x ≤ 0.)

Since F(∞) = $\int_{-\infty}^{\infty} f(y)dy = 1,$ f(x) is a legitimate pdf.

b. See above

c. P(2 ≤ X ≤ 5) = F(5) – F(2) = $1 - \frac{16}{81} - \left(1 - \frac{16}{36}\right) = .247$

(continued)

d. $E(x) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_{-\infty}^{\infty} x \cdot \frac{32}{(x+4)^3} dx = \int_0^{\infty} (x+4-4) \cdot \frac{32}{(x+4)^3} dx$

$= \int_0^{\infty} \frac{32}{(x+4)^2} dx - 4\int_0^{\infty} \frac{32}{(x+4)^3} dx = 8 - 4 = 4$

e. E(salvage value) = $= \int_0^{\infty} \frac{100}{x+4} \cdot \frac{32}{(y+4)^3} dx = 3200 \int_0^{\infty} \frac{1}{(y+4)^4} dx = \frac{3200}{(3)(64)} = 16.67$

95.

a. By differentiation,

$$f(x) = \begin{cases} x^2 & 0 \le x < 1 \\ \frac{7}{4} - \frac{3}{4}x & 1 \le y \le \frac{7}{3} \\ 0 & otherwise \end{cases}$$

b. P(.5 ≤ X ≤ 2) = F(2) – F(.5) = $1 - \frac{1}{2}\left(\frac{7}{3} - 2\right)\left(\frac{7}{4} - \frac{3}{4} \cdot 2\right) - \frac{(.5)^3}{3} = \frac{11}{12} = .917$

c. E(X) = $\int_0^1 x \cdot x^2 dx + \int_1^{7/3} x \cdot \left(\frac{7}{4} - \frac{3}{4}x\right) dx = \frac{131}{108} = 1.213$

96. μ = 40 V; σ = 1.5 V

a. P(39 < X < 42) = $\Phi\left(\frac{42-40}{1.5}\right) - \Phi\left(\frac{39-40}{1.5}\right)$

= Φ(1.33) - Φ(-.67) = .9082 - .2514 = .6568

b. We desire the 85th percentile: 40 + (1.04)(1.5) = 41.56

c. P(X > 42) = 1 – P(X ≤ 42) = 1 $- \Phi\left(\frac{42-40}{1.5}\right)$ = 1 - Φ(1.33) = .0918

Let D represent the number of diodes out of 4 with voltage exceeding 42.

P(D ≥ 1) = 1 – P(D = 0) = $1 - \binom{4}{0}(.0918)^0(.9082)^4$ =1 - .6803 = .3197

97. μ = 137.2 oz.; σ = 1.6 oz

a. P(X > 135) = $1-\Phi\left(\frac{135-137.2}{1.6}\right)$ = 1 - Φ(-1.38) = 1 - .0838 = .9162

b. With Y = the number among ten that contain more than 135 oz,
Y ~ Bin(10, .9162, so P(Y ≥ 8) = b(8; 10, .9162) + b(9; 10, .9162)
+ b(10; 10, .9162) =.9549.

c. μ = 137.2; $\frac{135-137.2}{s}=-1.65 \Rightarrow s=1.33$

98.

a. Let S = defective. Then p = P(S) = .05; n = 250 $\Rightarrow$ μ = np = 12.5, σ = 3.446. The random variable X = the number of defectives in the batch of 250. X ~ Binomial. Since np = 12.5 ≥ 10, and nq = 237.5 ≥ 10, we can use the normal approximation.
$P(X_{bin} \ge 25) \approx 1-\Phi\left(\frac{24.5-12.5}{3.446}\right)=1-\Phi(3.48)=1-.9997=.0003$

b. $P(X_{bin}=10) \approx P(X_{norm} \le 10.5) - P(X_{norm} \le 9.5)$
$=\Phi(-.58)-\Phi(-.87)=.2810-.1922=.0888$

99.

a. P(X > 100) = $1-\Phi\left(\frac{100-96}{14}\right)=1-\Phi(.29)=1-.6141=.3859$

b. P(50 < X < 80) = $\Phi\left(\frac{80-96}{14}\right)-\Phi\left(\frac{50-96}{14}\right)$
= Φ(-1.5) - Φ(-3.29) = .1271 - .0005 = .1266.

c. a = 5^{th} percentile = 96 + (-1.645)(14) = 72.97.
b = 95^{th} percentile = 96 + (1.645)(14) = 119.03. The interval (72.97, 119.03) contains the central 90% of all grain sizes.

100.

a. F(X) = 0 for x < 1 and = 1 for x > 3. For $1 \le x \le 3$, $F(x) = \int_{-\infty}^{x} f(y)dy$

$$= \int_{-\infty}^{1} 0dy + \int_{1}^{x} \frac{3}{2} \cdot \frac{1}{y^2} dy = 1.51\left(1 - \frac{1}{x}\right)$$

b. $P(X \le 2.5) = F(2.5) = 1.5(1 - .4) = .9$; $P(1.5 \le x \le 2.5) =$ $F(2.5) - F(1.5) = .4$

c. $E(X) = = \int_{1}^{3} x \cdot \frac{3}{2} \cdot \frac{1}{x^2} dx = \frac{3}{2} \int_{1}^{3} \frac{1}{x} dx = 1.5\ln(x)\Big]_1^3 = 1.648$

d. $E(X^2) = = \int_{1}^{3} x^2 \cdot \frac{3}{2} \cdot \frac{1}{x^2} dx = \frac{3}{2} \int_{1}^{3} dx = 3$, so $V(X) = E(X^2) - [E(X)]^2 = .284$,

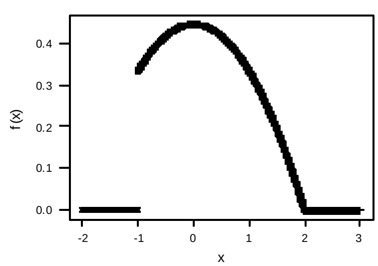
$\sigma = .553$

e. $h(x) = \begin{cases} 0 & 1 \le x \le 1.5 \\ x - 1.5 & 1.5 \le x \le 2.5 \\ 1 & 2.5 \le x \le 3 \end{cases}$

so $E[h(X)] = = \int_{1.5}^{2.5} (x - 1.5) \cdot \frac{3}{2} \cdot \frac{1}{x^2} dx + \int_{2.5}^{3} 1 \cdot \frac{3}{2} \cdot \frac{1}{x^2} dx = .267$

101.

a.



b. F(x) = 0 for x < -1 or == 1 for x > 2. For $-1 \le x \le 2$,

$$F(x) = \int_{-1}^{x} \frac{1}{9}(4 - y^2)dy = \frac{1}{9}\left(4x - \frac{x^3}{3}\right) + \frac{11}{27}$$

c. The median is 0 iff F(0) = .5. Since $F(0) = \frac{11}{27}$, this is not the case. Because $\frac{11}{27} < .5$, the median must be greater than 0.

d. Y is a binomial r.v. with n = 10 and $p = P(X > 1) = 1 - F(1) = \frac{5}{27}$

102.

a. E(X) = $\dfrac{1}{l}$ = 1.075, $s = \dfrac{1}{l}$ = 1.075

b. P(3.0 < X) = 1 – P(X ≤ 3.0) = 1 – F(3.0) = $3^{-.93(3.0)}$ = .0614
P(1.0 ≤ X ≤ 3.0) = F(3.0) – F(1.0) = .333

c. The 90th percentile is requested; denoting it by c, we have
.9 = F(c) = 1 – $e^{-(.93)c}$, whence c = $\dfrac{\ln(.1)}{(-.93)} = 2.476$

103.

a. P(X ≤ 150) = $\exp\left[-\exp\left(\dfrac{-(150-150)}{90}\right)\right] = \exp[-\exp(0)] = \exp(-1) = .368$, where exp(u) = e^u. P(X ≤ 300) = $\exp[-\exp(-1.6667)] = .828$,
and P(150 ≤ X ≤ 300) = .828 - .368 = .460.

b. The desired value c is the 90th percentile, so c satisfies
.9 = $\exp\left[-\exp\left(\dfrac{-(c-150)}{90}\right)\right]$. Taking the natural log of each side twice in succession yields ln[ln(.9)] = $\dfrac{-(c-150)}{90}$, so c = 90(2.250367) + 150 = 352.53.

c. f(x) = F′(X) = $\dfrac{1}{b}\cdot\exp\left[-\exp\left(\dfrac{-(x-a)}{b}\right)\right]\cdot\exp\left(\dfrac{-(x-a)}{b}\right)$

d. We wish the value of x for which f(x) is a maximum; this is the same as the value of x for which ln[f(x)] is a maximum. The equation of $\dfrac{d[\ln(f(x))]}{dx} = 0$ gives
$\exp\left(\dfrac{-(x-a)}{b}\right) = 1$, so $\dfrac{-(x-a)}{b} = 0$, which implies that x = α. Thus the mode is α.

e. E(X) = .5772β + α = 201.95, whereas the mode is 150 and the median is
–(90)ln[-ln(.5)] + 150 = 182.99. The distribution is positively skewed.

104.

a. E(cX) = cE(X) = $\dfrac{c}{l}$

b. E[c(1 - .5e^{ax})] = $\int_0^\infty c\left(1-.5e^{ax}\right)\cdot l\,e^{-lx}dx = \dfrac{c[.5l-a]}{l-a}$

105.

a. From a graph of f(x; μ, σ) or by differentiation, $x^* = \mu$.

b. No; the density function has constant height for $A \le X \le B$.

c. F(x; λ) is largest for x = 0 (the derivative at 0 does not exist since f is not continuous there) so $x^* = 0$.

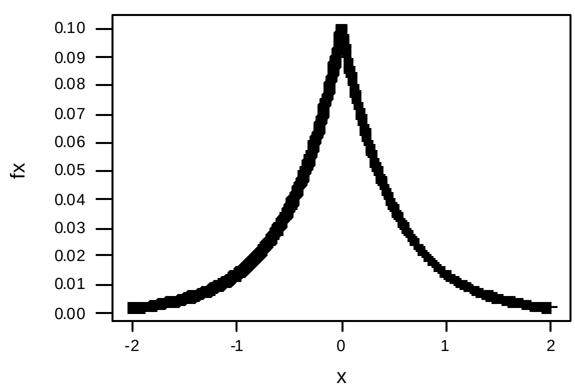
d. $\ln[f(x;a,b)] = -\ln(b^a) - \ln(\Gamma(a)) + (a-1)\ln(x) - \dfrac{x}{b};$

$$\frac{d}{dx}\ln[f(x;a,b)] = \frac{a-1}{x} - \frac{1}{b} \Rightarrow x = x^* = (a-1)b$$

e. From **d** $x^* = \left(\dfrac{n}{2} - 1\right)(2) = n - 2.$

106.

a. $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} .1e^{.2x}dx + \int_{0}^{\infty} .1e^{-.2x}dx = .5 + .5 = 1$



b. For x < 0, F(x) = $\int_{-\infty}^{x} .1e^{.2y}dy = \dfrac{1}{2}e^{.2x}$.

For x ≥ 0, F(x) = $\dfrac{1}{2} + \int_{0}^{x} .1e^{-.2y}dy = 1 - \dfrac{1}{2}e^{-.2x}$.

c. P(X < 0) = F(0) = $\dfrac{1}{2} = .5$, P(X < 2) = F(2) = 1 - .5$e^{-.4}$ = .665,

P(-1 ≤ X ≤ 2) – F(2) – F(-1) = .256, 1 - (-2 ≤ X ≤ 2) = .670

107.

a. Clearly f(x; λ_1, λ_2, p) ≥ 0 for all x, and $\int_{-\infty}^{\infty} f(x;\lambda_1,\lambda_2,p)dx$

$= \int_0^{\infty}\left[p\lambda_1 e^{-\lambda_1 x} + (1-p)\lambda_2 e^{-\lambda_2 x}\right]dx = p\int_0^{\infty}\lambda_1 e^{-\lambda_1 x}dx + (1-p)\int_0^{\infty}\lambda_2 e^{-\lambda_2 x}dx$

= p + (1 – p) = 1

b. For x > 0, F(x; λ_1, λ_2, p) = $\int_0^{x} f(y;\lambda_1,\lambda_2,p)dy = p(1-e^{-\lambda_1 x}) + (1-p)(1-e^{-\lambda_2 x})$.

c. E(X) = $\int_0^{\infty} x\cdot\left[p\lambda_1 e^{-\lambda_1 x}) + (1-p)\lambda_2 e^{-\lambda_2 x})\right]dx$

$$= p\int_0^{\infty} x\lambda_1 e^{-\lambda_1 x}dx + (1-p)\int_0^{\infty} x\lambda_2 e^{-\lambda_2 x}dx = \frac{p}{\lambda_1} + \frac{(1-p)}{\lambda_2}$$

d. E(X^2) = $\dfrac{2p}{\lambda_1^2} + \dfrac{2(1-p)}{\lambda_2^2}$, so Var(X) = $\dfrac{2p}{\lambda_1^2} + \dfrac{2(1-p)}{\lambda_2^2} - \left[\dfrac{p}{\lambda_1} + \dfrac{(1-p)}{\lambda_2}\right]^2$

e. For an exponential r.v., CV = $\dfrac{1/\lambda}{1/\lambda} = 1$. For X hyperexponential,

$$\text{CV} = \left[\frac{\frac{2p}{\lambda_1^2} + \frac{2(1-p)}{\lambda_2^2}}{\left[\frac{p}{\lambda_1} + \frac{(1-p)}{\lambda_2}\right]^2} - 1\right]^{1/2} = \left[\frac{2\left(p\lambda_2^2 + (1-p)\lambda_1^2\right)}{\left(p\lambda_2 + (1-p)\lambda_1\right)^2} - 1\right]^{1/2}$$

= $[2r-1]^{1/2}$ where r = $\dfrac{\left(p\lambda_2^2 + (1-p)\lambda_1^2\right)}{\left(p\lambda_2 + (1-p)\lambda_1\right)^2}$. But straightforward algebra shows that r > 1 provided $\lambda_1 \neq \lambda_2$, so that CV > 1.

f. $\mu = \dfrac{n}{\lambda}$, $\sigma^2 = \dfrac{n}{\lambda^2}$, so $\sigma = \dfrac{\sqrt{n}}{\lambda}$ and CV = $\dfrac{1}{\sqrt{n}} < 1$ if n > 1.

108.

a. $1 = \int_5^{\infty} \frac{k}{x^a}\,dx = k \cdot \frac{5^{1-a}}{a-1} \Rightarrow k = (a-1)5^{1-a}$ where we must have α > 1.

b. For x ≥ 5, F(x) = $\int_5^{x} \frac{k}{y^a}\,dy = 5^{1-a}\left[\frac{1}{5^{1-a}} - \frac{1}{x^{a-1}}\right] = 1 - \left(\frac{5}{x}\right)^{a-1}$.

c. E(X) = $\int_5^{\infty} x \cdot \frac{k}{x^a}\,dx = \int_5^{\infty} x \cdot \frac{k}{x^{a-1}}\,dx = \frac{k}{5^{a-2} \cdot (a-2)}$, provided α > 2.

d. $P\left(\ln\left(\frac{X}{5}\right) \le y\right) = P\left(\frac{X}{5} \le e^y\right) = P\left(X \le 5e^y\right) = F\left(5e^y\right) = 1 - \left(\frac{5}{5e^y}\right)^{a-1}$

$1 - e^{-(a-1)y}$, the cdf of an exponential r.v. with parameter α - 1.

109.

a. A lognormal distribution, since $\ln\left(\frac{I_o}{I_i}\right)$ is a normal r.v.

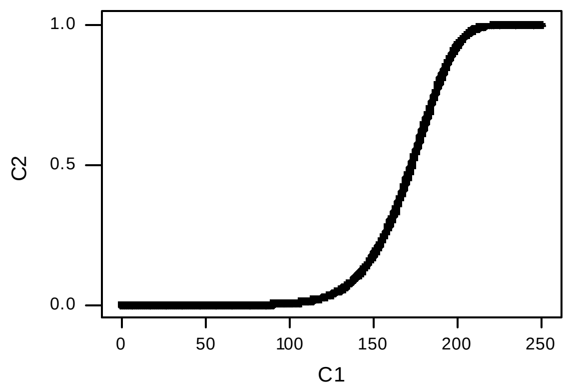
b. $P(I_o > 2I_i) = P\left(\frac{I_o}{I_i} > 2\right) = P\left(\ln\left(\frac{I_o}{I_i}\right) > \ln 2\right) = 1 - P\left(\ln\left(\frac{I_o}{I_i}\right) \le \ln 2\right)$

$1 - \Phi\left(\frac{\ln 2 - 1}{.05}\right) = 1 - \Phi(-6.14) = 1$

c. $E\left(\frac{I_o}{I_i}\right) = e^{1+.0025/2} = 2.72,\ Var\left(\frac{I_o}{I_i}\right) = e^{2+.0025} \cdot \left(e^{.0025} - 1\right) = .0185$

110.

a.



b. P(X > 175) = 1 – F(175; 9, 180) = $e^{-\left(\frac{175}{180}\right)^9} = .4602$
P(150 ≤ X ≤ 175) = F(175; 9, 180) - F(150; 9, 180)
= .5398 - .1762 = .3636

c. P(at least one) = 1 – P(none) = 1 – (1 - .3636)^2 = .5950

d. We want the 10th percentile: .10 = F(x; 9, 180) = $1 - e^{-\left(\frac{x}{180}\right)^9}$. A small bit of algebra leads us to x = 140.178. Thus 10% of all tensile strengths will be less than 140.178 MPa.

111. F(y) = P(Y ≤ y) = P(σZ + μ ≤ y) = $P\left(Z \le \frac{(y-m)}{s}\right) = \int_{-\infty}^{\frac{(y-m)}{s}} \frac{1}{\sqrt{2p}} e^{-\frac{1}{2}z^2}\,dz$. Now differentiate with respect to y to obtain a normal pdf with parameters μ and σ.

112.

a. $F_Y(y)$ = P(Y ≤ y) = P(60X ≤ y) = $P\left(X \le \frac{y}{60}\right) = F\left(\frac{y}{60b}; a\right)$. Thus $f_Y(y)$

$= f\left(\frac{y}{60b}; a\right) \cdot \frac{1}{60b} = \frac{y^{a-1} e^{\frac{-y}{60b}}}{(60b)^a\,\Gamma(a)}$, which shows that Y has a gamma distribution with parameters α and 60β.

b. With c replacing 60 in **a**, the same argument shows that cX has a gamma distribution with parameters α and cβ.

113.

a. Y = -ln(X) $\Rightarrow$ x = e^{-y} = k(y), so k′(y) = $-e^{-y}$. Thus since f(x) = 1, g(y) = 1 · | $-e^{-y}$ | = e^{-y} for 0 < y < ∞, so y has an exponential distribution with parameter λ = 1.

b. y = σZ + μ $\Rightarrow$ y = h(z) = σZ + μ $\Rightarrow$ z = k(y) = $\dfrac{(y-m)}{s}$ and k′(y) = $\dfrac{1}{s}$, from which the result follows easily.

c. y = h(x) = cx $\Rightarrow$ x = k(y) = $\dfrac{y}{c}$ and k′(y) = $\dfrac{1}{c}$, from which the result follows easily.

114.

a. If we let $a = 2$ and $b = \sqrt{2}s$, then we can manipulate f(v) as follows:

$$f(n) = \frac{n}{s^2} e^{-n^2/2s^2} = \frac{2}{2s^2} n e^{-n^2/2s^2} = \frac{2}{\left(\sqrt{2}s\right)^2} n^{2-1} e^{-\left(n/\sqrt{2}s\right)^2} = \frac{a}{b^a} n^{a-1} e^{-\left(n/b\right)^a},$$

which is in the Weibull family of distributions.

b. $F(n) = \int_0^{25} \frac{n}{400} e^{\frac{-n}{800}} dn$; cdf: $F\left(n;2,\sqrt{2}s\right) = 1 - e^{-\left(\frac{n}{\sqrt{2}s}\right)} = 1 - e^{\frac{-v^2}{800}}$, so

$$F\left(25;2,\sqrt{2}\right) = 1 - e^{\frac{-625}{800}} = 1 - .458 = .542$$

115.

a. Assuming independence, P(all 3 births occur on March 11) = $\left(\frac{1}{365}\right)^3 = .00000002$

b. $\left(\frac{1}{365}\right)^3 (365) = .0000073$

c. Let X = deviation from due date. X~N(0, 19.88). Then the baby due on March 15 was 4 days early. P(x = -4) ˜ P(-4.5 < x < -3.5)

$$= \Phi\left(\frac{-3.5}{19.88}\right) - \Phi\left(\frac{-4.5}{19.88}\right) = \Phi(-.18) - \Phi(-.237) = .4286 - .4090 = .0196 .$$

Similarly, the baby due on April 1 was 21 days early, and P(x = -21)

$$˜\ \Phi\left(\frac{-20.5}{19.88}\right) - \Phi\left(\frac{-21.5}{19.88}\right) = \Phi(-1.03) - \Phi(-1.08) = .1515 - .1401 = .0114 .$$

The baby due on April 4 was 24 days early, and P(x = -24) ˜ .0097

Again, assuming independence, P(all 3 births occurred on March 11) =

$$(.0196)(.0114)(.0097) = .00002145$$

d. To calculate the probability of the three births happening on any day, we could make similar calculations as in part c for each possible day, and then add the probabilities.

116.

a. F(x) = le^{-lx} and F(x) = $1-e^{-lx}$, so r(x) = $\dfrac{le^{-lx}}{e^{-lx}} = l$, a constant (independent of X); this is consistent with the memoryless property of the exponential distribution.

b. r(x) = $\left(\dfrac{a}{b^{a}}\right)x^{a-1}$; for α > 1 this is increasing, while for α < 1 it is a decreasing function.

c. ln(1 – F(x)) = $-\int a\left(1-\dfrac{x}{b}\right)dx = -a\left[x-\dfrac{x^2}{2b}\right] \Rightarrow F(x) = 1-e^{-a\left(x-\frac{x^2}{2b}\right)}$,

f(x) = $a\left(1-\dfrac{x}{b}\right)e^{-a\left(x-\frac{x^2}{2b}\right)}$ $\qquad 0 \le x \le \beta$

117.

a. $F_X(x)$ = $P\left(-\dfrac{1}{l}\ln(1-U) \le x\right) = P(\ln(1-U) \ge -lx) = P(1-U \ge e^{-lx})$

$= P(U \le 1-e^{-lx}) = 1-e^{-lx}$ since $F_U(u)$ = u (U is uniform on [0, 1]). Thus X has an exponential distribution with parameter λ.

b. By taking successive random numbers u_1, u_2, u_3, …and computing $x_i = -\dfrac{1}{10}\ln(1-u_i)$, … we obtain a sequence of values generated from an exponential distribution with parameter λ = 10.

118.

a. E(g(X)) ≈ E[g(μ) + g′(μ)(X - μ)] = E(g(μ)) + g′(μ)·E(X - μ), but E(X) - μ = 0 and E(g(μ)) = g(μ) (since g(μ) is constant), giving E(g(X)) ≈ g(μ).
V(g(X)) ≈ V[g(μ) + g′(μ)(X - μ)] = V[g′(μ)(X - μ)] = $(g'(\mu))^2$·V(X - μ) = $(g'(\mu))^2$·V(X).

b. $g(I) = \dfrac{v}{I}, g'(I) = \dfrac{-v}{I^2}$, so $E(g(I)) = m_R \approx \dfrac{v}{m_I} = \dfrac{v}{20}$

$V(g(I)) \approx \left(\dfrac{-v}{m_I^2}\right)^2 \cdot V(I), s_{g(I)} \approx \dfrac{v}{20^2} \cdot s_I = \dfrac{v}{800}$

119. g(μ) + g′(μ)(X - μ) ≤ g(X) implies that E[g(μ) + g′(μ)(X - μ)] = E(g(μ)) = g(μ) ≤ E(g(X)), i.e. that g(E(X)) ≤ E(g(X)).

120. For $y > 0$, $F(y) = P(Y \le y) = P\left(\frac{2X^2}{b^2} \le y\right) = P\left(X^2 \le \frac{b^2 y}{2}\right) = P\left(X \le \frac{b\sqrt{y}}{\sqrt{2}}\right)$. Now take the cdf of X (Weibull), replace x by $\frac{b\sqrt{y}}{\sqrt{2}}$, and then differentiate with respect to y to obtain the desired result $f_Y(y)$.